

J -Trace Identities and Invariant Theory

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Abstract

We generalize the notion of trace identity to J -trace. Our main result is that all J -trace identities of $M_{n,n}$ are consequence of those of degree $\frac{1}{2}n(n+3)$. This also gives an indirect description of the queer trace identities of $M_n(E)$.

Keywords— Trace identities, J -trace-identities, queer trace identities, invariant theory, Sergeev algebra

1 Introduction

1.1 Traces, Supertraces and J -Traces

Let F be a field. A trace function on an F -algebra A is a linear map from A to a commutative F -algebra, typically the center of A , such that $tr(ab) = tr(ba)$ for all $a, b \in A$. Even if the image of the trace map is not in the center of A , we will always take it to be in a commutative algebra that acts on A . One generalization that has been studied is that in which A is a \mathbb{Z}_2 -graded algebra, or superalgebra, $A = A_0 \oplus A_1$. In this case one studies supertraces, which are linear, homogeneous (degree preserving) maps from A to a supercommutative ring which has a graded action on A such that $str(ab) = (-1)^{\alpha\beta} str(ba)$ for all homogeneous $a \in A_\alpha, b \in A_\beta$. By the homogeneity of the supertrace we also have $a \cdot str(b) = (-1)^{\alpha\beta} str(b)a$. A basic example is $A = M_n(E)$, where E is the infinite Grassmann algebra with its usual grading making it into a supercommutative algebra, and $A_i =$

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$M_n(E_i)$ for $i \in \mathbb{Z}_2$. If $a = (a_{ij}) \in M_n(E)$ then the supertrace of can be defined as $str(a) = \sum a_{ii}$.

There are other gradings and supertraces defined on $M_n(E)$. Let ω be the involution on E defined by $\omega(e) = (-1)^\alpha e$ for all $e \in E_\alpha$. Given $k+\ell = n$, we embed E into $M_n(E)$ by mapping each e to the diagonal matrix

$$\omega_{k,\ell}(e) = \text{diag}(\underbrace{e, \dots, e}_k, \underbrace{\omega(e), \dots, \omega(e)}_\ell),$$

and define $M(k, \ell)_0$ to be the elements of $M_n(E)$ which commute with E , and $M(k, \ell)_1$ to be the ones that supercommute. It is not hard to see that $M(k, \ell)_0$ consists of all $n \times n$ matrices of the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in which $A \in M_k(E_0)$, $D \in M_\ell(E_0)$, and B and C have entries in E_1 ; and $M(k, \ell)_1$ is the opposite. The algebra $M(k, \ell)_0$ is generally denoted $M_{k,\ell}$. The supertrace on $M(k, \ell)$ is defined on homogeneous elements of degree d as

$$str(a_{ij}) = \sum_{i=1}^k a_{ii} - (-1)^d \sum_{i=k+1}^n a_{ii},$$

and extended to all of $M(k, \ell)$ by linearity. It is useful to consider $M(k, \ell)$ to be an E -algebra by letting each $e \in E$ act as $\omega_{k,\ell}(e)$. With this action we have that the supertrace is E -linear in the natural sense that $str(ea) = e \cdot str(a)$ for all a .

In this paper we will be interested in algebras A with supertrace with a degree one element j such $j^2 = 1$. The principal example is $M(n, n)$ and $j = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. For any such algebra $A_1 = jA_0 = A_0j$ and $A_0 = jA_1 = A_1j$. Moreover, if $a \in A_0$ commutes with j then

$$str(a) = str(j^2a) = -str(jaj) = -str(a)$$

and unless the characteristic is 2 this implies $str(a) = 0$.

1.2 Universal Algebras

Let X be an infinite set. Define $\overline{F}\langle X, tr \rangle$ to be the commutative F -algebra generated by the symbols $tr(u)$ for u in the free algebra $F\langle X \rangle$ with the relations $tr(\alpha u) = \alpha tr(u)$, $tr(u + v) = tr(u) + tr(v)$ and $tr(uv) = tr(vu)$ for all $\alpha \in F$, $u, v \in F\langle X \rangle$. There is an obvious trace map from $F\langle X \rangle$ to $\overline{F}\langle X, tr \rangle$ with the universal property that given any algebra homomorphism $F\langle X \rangle \rightarrow A$, where A is an algebra with trace $A \rightarrow C$, there is a unique map $\overline{F}\langle X, tr \rangle \rightarrow C$ making the following diagram commute:

$$\begin{array}{ccc}
F\langle X \rangle & \longrightarrow & A \\
\downarrow & & \downarrow \\
\overline{F}\langle X, tr \rangle & \longrightarrow & C
\end{array}$$

Let $F\langle X, tr \rangle$ be the tensor product $\overline{F}\langle X, tr \rangle \otimes F\langle X \rangle$. There are two traditional approaches to defining a trace function on $F\langle X, tr \rangle$, the issue being how to define $tr(tr(u)) = tr(u)tr(1)$. Either one defines a new commuting variable $\gamma = tr(1)$, or else chooses an integer n and sets $tr(1) = n$ in order to study algebras with $tr(1) = n$. At any rate, if A is an F -algebra with trace function to the central subalgebra C , (and with $tr(1) = n$ if we are in the latter case), then the above universal property takes the simpler form that given any set theoretic map $X \rightarrow A$ there is a unique extension to a homomorphism $F\langle X, tr \rangle \rightarrow A$ which preserves trace. Elements of $F\langle X, tr \rangle$ are called trace polynomials or mixed trace polynomials, and ones which are zero under every trace preserving homomorphism $F\langle X, tr \rangle \rightarrow A$, for a given A , are called trace identities for A ; and elements of $\overline{F}\langle X, tr \rangle$ are called pure trace polynomials and ones that are identities for some A are called pure trace identities of A . The set of trace identities for A will be denoted $I(A, tr)$ and it is an ideal of $F\langle X, tr \rangle$. The quotient will be denoted $U(A, tr)$ and it has the expected universal properties. Likewise $\bar{I}(A, tr) = I(A, tr) \cap \overline{F}\langle X, tr \rangle$ and $\bar{U}(A, tr) = \overline{F}\langle X, tr \rangle / \bar{I}(A, tr)$.

Given two sets of variables X and Y , considered degree 0 and degree 1, respectively, there are easy generalizations of these concepts to $\overline{F}\langle X, Y, str \rangle$ and $F\langle X, Y, str \rangle$, the free superalgebra with supertrace, including the notations of supertrace identities or pure supertrace identities. Note that $\overline{F}\langle X, Y, str \rangle$ is required to be supercentral in $F\langle X, Y, str \rangle$ so that if $u \in \overline{F}\langle X, Y, str \rangle$ is degree one, then it anticommutes with all degree one elements. For example, the Grassmann algebra has a supertrace map $str : E \rightarrow E$ given by the function ω defined above. With respect to this supertrace E satisfies the mixed supertrace identities

$$str(x) = x \text{ and } str(y) = -y$$

and all supertrace identities of E are consequences of these two.

For our purposes we will need another generalization. We will be considering superalgebras with supertrace, which have a degree one element j whose square is equal to 1. In such a case we say that the algebra has a J -trace.

We define $F\langle X, J\text{-trace} \rangle$ to be the free supertrace algebra $F\langle X, \{J\}, str \rangle$ modulo the relation that $J^2 = 1$. Note that all variables are even and

J is odd. Elements of $F\langle X, J\text{-trace} \rangle$ are called J -trace polynomials, and elements of $\bar{F}\langle X, \{J\}, J\text{-trace} \rangle$ are called pure J -trace polynomials. And, for algebras A with supertrace and designated odd idempotent j , one defines $I(A, J\text{-trace})$ as the ideal of J -trace identities for A and $U(A, J\text{-trace})$ the corresponding universal algebra. Likewise, $\bar{I}(A, J\text{-trace})$ will be the pure J -trace identities, and $\bar{U}(A, J\text{-trace})$ will be the quotient $\bar{F}\langle X, J\text{-trace} \rangle / \bar{I}(A, J\text{-trace})$. For example, $M(1, 1)$ satisfies the mixed J -trace identity

$$2[Jx_1 + x_1J, Jx_2 + x_2J] = \text{str}(Jx_1 + x_1J)\text{str}(Jx_2 + x_2J) \quad (1)$$

From one point of view J -trace identities are identities of A_0 since each variable x_i is degree zero and substituted by elements of A_0 only. For this reason we may speak of the J -trace of A_0 . On the other hand, if we wanted to describe a graded J -trace identity $f(x_1, \dots, x_k, y_1, \dots, y_\ell)$ in both even and odd variables we could simply let

$$g(x_1, \dots, x_{k+\ell}) = f(x_1, \dots, x_k, Jx_{k+1}, \dots, Jx_{k+\ell}).$$

Then f will vanish under all graded substitutions $x_i \mapsto a_i \in A_0$, $y_j \mapsto a_j \in A_1$ if and only if g vanishes under all substitutions $x_i \mapsto a_i \in A_0$.

1.3 J -trace-identities of $M_{n,n}$

Our main result is the determination of the J -trace-identities of $M(n, n)$ in characteristic zero. Following in the footsteps of Procesi, see [6], our approach is based on invariant theory. Procesi's work in [6] is based on the invariant theory of the general linear group which in turn is based on the double centralizer theorem between the general linear group and the symmetric group. The progression of ideas is similar here. In [9] Sergeev proved a double centralizer theorem for the queer superalgebra; and in [2] we used Sergeev's theory to study the invariant theory of the queer superalgebra. We proved that the pure J -trace polynomials in d variables give all maps $M(n, n)^d \rightarrow E$ invariant under the action of the queer superalgebra. In the language of invariant theory this theorem could be called the First Fundamental Theorem for the queer superalgebra and in the same spirit the current work, giving the relations between the J -trace polynomials would be the Second Fundamental Theorem.

In section 2 of the current work we show how to associate multilinear pure J -trace-polynomials with elements of the Sergeev algebra and we show that this association has properties similar to Razmyslov and Procesi's association of trace polynomials with the group algebra of the symmetric group.

(For the reader familiar with polynomial identities: But not analogous to Regev's association of ordinary multilinear polynomials with elements of the group algebra of the symmetric group.) Then in section 3 we describe which elements of the Sergeev algebra correspond to J -trace-polynomial identities for $M(n, n)$ and prove our main theorem, that in characteristic zero all such identities are consequences of the ones of degree $\binom{n+2}{2}$. It then follows from the non-degeneracy of the J -trace that all mixed J -trace-identities are consequences of the ones of degree $\binom{n+2}{2} - 1 = \frac{1}{2}n(n+3)$.

1.4 Queer Traces

Readers familiar with Kemer's structure theory of p.i. algebras (see [4]) will know that it is based on the three families of verbally prime algebras, $M_n(F)$, $M_{k,\ell}$, and $M_n(E)$. Although the polynomial identities are not well understood for any of the families, the trace identities are known for the first two: The case of trace identities of matrices was done by Razmyslov and Procesi, see [7] and [6], and the case of $M_{k,\ell}$ by Razmyslov and Berele, see [8] and [1]. The third family $M_n(E)$ is not graded with a natural trace, but it does have a queer trace. The queer trace of a matrix with Grassmann entries is defined to be the degree one part of the sum of the diagonal entries. This map is similar to a trace function in that it satisfies $qtr(ab) = qtr(ba)$, but unlike ordinary traces, queer trace anticommute, $qtr(a)qtr(b) = -qtr(b)qtr(a)$. In this section we show how the theory of J -trace-identities of $M(n, n)$ can be used to describe the queer trace identities of $M_n(E)$.

In general there are two ways to construct algebras with queer trace from graded algebras with supertrace. If A is such an algebra and if $str(A_0) = 0$ then it is not hard to see that $str(ab) = str(ba)$ and $str(a)str(b) = -str(b)str(a)$ and so the supertrace is a queer trace. So, given any superalgebra A with supertrace, we can simply define $qtr(a)$ to be the degree one part of $str(a)$. This is how the queer trace on $M_n(E)$ was defined. The second construction can be used if in addition A had an idempotent element $j \in A_1$. In this case we could let B be the centralizer of j . B would be a graded superalgebra with $str(B_0) = 0$ and so the supertrace on B would be a queer trace. For our purposes we will need the case of $A = M(n, n)$. The centralizer of j is the set of matrices of the form $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$. This algebra is denoted $SM(n, n)$, and the degree zero part will be denoted $SM_{n,n}$. (S for symmetric.)

There are technical issues in defining mixed queer trace identities since queer traces are not central. Pure queer trace polynomials can easily be defined as the free algebra on the symbols $qtr(u)$ where u is a monomial modulo

the obvious relations. As an example, the Grassmann algebra satisfies the mixed queer trace identity

$$[x_1, x_2] = 2qtr(x_1)qtr(x_2) \quad (2)$$

and the pure queer trace identities

$$qtr(x_1[x_2, x_3]) = qtr(x_1)qtr(x_2)qtr(x_3), \quad (3)$$

$$qtr(x_1)qtr(x_2x_3) + qtr(x_2)qtr(x_1x_3) + qtr(x_3)qtr(x_1x_2) = 0. \quad (4)$$

The reader may wish to show that (3) and (4) are each consequences of (2). The intrepid reader may wish to show that all queer trace identities (appropriately defined) of E are consequences of (2), and that all pure queer trace identities are consequences of (3) and (4).

There is an isomorphism from $SM_{n,n}$ to $M_n(E)$ given by

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \mapsto A + B.$$

It is desirable to make this map queer trace preserving and to that end we define \widetilde{qtr} on $SM_{n,n}$ as

$$\widetilde{qtr} \begin{pmatrix} A & B \\ B & A \end{pmatrix} = tr(B),$$

or, more naturally, as $\widetilde{qtr}(a) = \frac{1}{2}str(ja)$,

Since we can describe all J -trace-identities of $M(n, n)$ the following theorem will describe all queer trace identities of $M_n(E)$. If A is J -trace-algebra then we make the universal algebra $U(A, J\text{-trace})$ into an algebra with queer trace by modding out by the relation that J is central. Denote by Q the map

$$Q : \overline{F}\langle X, J\text{-trace} \rangle \rightarrow \overline{F}\langle X, qtr \rangle,$$

from supertrace polynomials to queer trace polynomials gotten by making J central and identifying $str(ja)$ with $2qtr(a)$. Note that when we make J central we make supertraces of even terms 0.

Theorem 1.1. *In characteristic zero, the ideal of queer trace identities of $M_n(E)$, equals the ideal of J -trace-identities of $M(n, n)$ together with the identity that J is central.*

Proof. Since $M_n(E)$ is isomorphic to $SM_{n,n}$ as an algebra with queer trace, we consider queer trace identities of $SM_{n,n}$. Also, since $SM_{n,n}$ is the centralizer of j in $M_{n,n}$, all supertrace of identities of $M_{n,n}$ become queer trace identities of $SM_{n,n}$ under the map Q . Since all identities are consequences of multilinear ones, in order to prove the theorem it suffices to prove that if $f(x_1, \dots, x_k)$ is a multilinear queer trace identity for $SM_{n,n}$ there exists a J -trace polynomial $g(x_1, \dots, x_k)$ such that $Q(g) = f$. First, let $f_1(x_1, \dots, x_k)$ be any supertrace polynomial such that $Q(f_1) = f$. Then, f_1 will be a J -trace-identity for $SM_{n,n}$, and so we let

$$g(x_1, \dots, x_k) = f_1(x_1 + Jx_1J, \dots, x_k + Jx_kJ).$$

For each degree zero element a of $M_{n,n}$ $a + jaj$ commutes with j and so is a degree zero element of $SM_{n,n}$. Hence, g is a J -trace-identity for $M_{n,n}$ and $Q(g) = f(2x_1, \dots, 2x_k) = 2^k f(x_1, \dots, x_k)$ for all $x_1, \dots, x_k \in SM_{n,n}$. \square

2 Properties of Multilinear J -Trace Identities

2.1 The Sergeev algebra

Given a positive integer d the Clifford algebra $C(d)$ is the 2^d -dimensional algebra generated by the anticommuting idempotents c_1, \dots, c_d . The symmetric group S_d acts on $C(d)$, and the Segeev algebra $W(d)$ is the semidirect product of FS_d with $C(d)$. More concretely, $S(d)$ is spanned by all $\sigma c_1^{\epsilon_1} \cdots c_d^{\epsilon_d}$ with relations

$$c_i^2 = 1, \quad c_i c_j = -c_j c_i, \quad \sigma c_i = c_{\sigma(i)} \sigma.$$

In keeping with [2] we will be using the graded opposite of $W(d)$ instead of $W(d)$. It is generated by permutations σ and by Clifford elements C_i with relations

$$C_i^2 = 1, \quad C_i C_j = -C_j C_i, \quad C_i \sigma = \sigma C_{\sigma(i)}, \quad \sigma \cdot \tau = \tau \sigma \quad (5)$$

for $i, j = 1, \dots, d$ and $\sigma, \tau \in S_d$. In the last relation the dot is for the product in $W(d)^o$ and the juxtaposition $\tau \sigma$ indicates the ordinary product in S_d .

There is a well-known identification of elements of FS_d with multilinear, degree d pure trace polynomials. If σ has cycle decomposition

$$\sigma = (i_1, \dots, i_a) \cdots (j_1, \dots, j_b)$$

then $tr_\sigma(x_1, \dots, x_d)$ is defined to be

$$tr(x_{i_1} \cdots x_{i_a}) \cdots tr(x_{j_1} \cdots x_{j_b}). \quad (6)$$

Note that this is well-defined because cycles commute and traces commute, and elements can be permuted cyclically in both traces and cycles. Another nice property of this identification is that

$$tr_{\sigma\tau\sigma^{-1}}(x_1, \dots, x_d) = tr_{\tau}(x_{\sigma(1)}, \dots, x_{\sigma(d)}) \quad (7)$$

Hypothetically, one would like to identify elements of the Sergeev algebra with multilinear J -trace polynomials via

$$\sigma C_1^{\epsilon_1} \dots C_d^{\epsilon_d} \rightarrow tr_{\sigma}(J^{\epsilon_1} x_1, \dots, J^{\epsilon_d} x_d).$$

Unfortunately, such an identification would not be well-defined. Two pure J -traces may commute or anticommute: They will anticommute if they are each of odd degree in J . Likewise, permuting the elements within a trace may result in a negative sign. On the other hand, for given $\sigma \in W(d)$ there are only two different J -trace polynomials that could result from the above putative identification and they are negatives of each other, and so we need to choose one in a natural way.

To identify elements of the Sergeev algebra with J -trace polynomials, we first define str_{σ} for permutations. Assume that some of the variables x_1, \dots, x_d are even and the remainder odd, and let e_1, \dots, e_d be supercentral variables with $\deg x_i = \deg e_i$ for all i , and such that $str(e_i u) = e_i str(u)$ for all u . Then $str_{\sigma}(x_1, \dots, x_d)$ is uniquely determined by the condition

$$tr_{\sigma}(e_1 x_1, \dots, e_d x_d) = e_d \dots e_1 str_{\sigma}(x_1, \dots, x_d).$$

If $w = \sigma C_1^{\epsilon_1} \dots C_d^{\epsilon_d} \in W(d)^o$ we define

$$jtr_w(x_1, \dots, x_d) = str_{\sigma}(J^{\epsilon_1} x_1, \dots, J^{\epsilon_d} x_d),$$

recalling that the x_i are all even and J is odd. Combining the previous two equations we get

$$e_d \dots e_1 jtr_w(x_1, \dots, x_d) = tr_{\sigma}(e_1 J^{\epsilon_1} x_1, \dots, e_d J^{\epsilon_d} x_d), \quad (8)$$

which can be taken as the definition of jtr_w when w is a monomial. Finally, using linearity we can define jtr_w for any $w \in W(d)^o$.

More generally, if $w = \sigma C_{i_1} \dots C_{i_k}$ with the i_a distinct, then jtr_w is gotten from $tr_{\sigma}(x_1, \dots, x_d)$ by substituting Jx_{i_a} for each x_{i_a} and pulling out a factor of $e_{i_k} \dots e_{i_1}$ on the left, whether the sequence of i_a is increasing or not.

2.2 Conjugation and J -trace Polynomials

The goal of this section is to generalize (7) to J -trace polynomials. There are two generalizations, one deals with conjugation by a permutation and the other deals with conjugation by a generator of the Clifford algebra.

Lemma 2.1. *Let w be an element of $W(d)^o$ and let $\sigma \in S_d \subseteq W(d)^o$ be a permutation. Then $jtr_{\sigma w \sigma^{-1}}(x_1, \dots, x_d) = jtr_w(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(d)})$.*

As the proof will show, there is a σ^{-1} instead of a σ in the formula unlike (7) because we are using the opposite of S_d .

Proof. Without loss, we may take $w = \tau C_1^{\epsilon_1} \dots C_d^{\epsilon_d} = \tau c$ and so by (5)

$$\sigma w \sigma^{-1} = \sigma \cdot \tau \cdot \sigma^{-1} C_{\sigma^{-1}(1)}^{\epsilon_1} \dots C_{\sigma^{-1}(d)}^{\epsilon_d} = \sigma^{-1} \tau \sigma C_{\sigma^{-1}(1)}^{\epsilon_1} \dots C_{\sigma^{-1}(d)}^{\epsilon_d}.$$

Then

$$C_{\sigma^{-1}(1)}^{\epsilon_1} \dots C_{\sigma^{-1}(d)}^{\epsilon_d} = g C_1^{\epsilon_{\sigma(1)}} \dots C_d^{\epsilon_{\sigma(d)}},$$

where $g = \pm 1$. Let e_i be supercentral with degrees equal to $\epsilon_{\sigma(i)}$. Then

$$e_d \dots e_1 jtr_{\sigma w \sigma^{-1}}(x_1, \dots, x_d) = g \cdot tr_{\sigma^{-1} \tau \sigma}(e_1 J^{\epsilon_{\sigma(1)}} x_1, \dots, e_d J^{\epsilon_{\sigma(d)}} x_d).$$

By (7) this equals

$$\begin{aligned} & g \cdot tr_{\tau}(e_{\sigma^{-1}(1)} J^{\epsilon_1} x_{\sigma^{-1}(1)}, \dots, e_{\sigma^{-1}(d)} J^{\epsilon_d} x_{\sigma^{-1}(d)}) \\ &= g \cdot e_{\sigma^{-1}(d)} \dots e_{\sigma^{-1}(1)} jtr_w(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(d)}) \\ &= \pm e_d \dots e_1 jtr_w(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(d)}). \end{aligned}$$

To complete the proof we need to check that the sign is positive. Since the degree of e_i equals the degree of $\epsilon_{\sigma(i)}$, the definition of g implies that

$$g e_1 \dots e_d = e_{\sigma^{-1}(1)} \dots e_{\sigma^{-1}(d)}.$$

This is not exactly what we want because the indices are increasing rather than decreasing. We separate the last step of the proof into a sublemma.

Sublemma. Let e_1, \dots, e_d be homogeneous elements of the Grassmann algebra E and let $\sigma \in S_d$ be a permutation such that $g e_1 \dots e_d = e_{\sigma^{-1}(1)} \dots e_{\sigma^{-1}(d)}$. Then $g e_d \dots e_1 = e_{\sigma^{-1}(d)} \dots e_{\sigma^{-1}(1)}$.

Proof of Sublemma: Let E be the Grassmann algebra on the vector space V with basis v_1, v_2, \dots and assume without loss of generality that each e_i is a monomial in the v_j . There is a antiisomorphism θ on E given by

$$v_{i_1} \cdots v_{i_t} \mapsto v_{i_t} \cdots v_{i_1}.$$

For each monomial e_i we have $\theta(e_i) = g_i e_i$ where $g_i = \pm 1$. Now,

$$\begin{aligned} \theta(e_d \cdots e_1) &= \theta(e_1) \cdots \theta(e_d) \\ &= g_1 \cdots g_d e_1 \cdots e_d \\ &= g g_1 \cdots g_d e_{\sigma^{-1}(1)} \cdots e_{\sigma^{-1}(d)} \\ &= g \theta(e_{\sigma^{-1}(1)}) \cdots \theta(e_{\sigma^{-1}(d)}) \\ &= g \theta(e_{\sigma^{-1}(d)} \cdots e_{\sigma^{-1}(1)}). \end{aligned}$$

Since θ is a one-to-one linear transformation the sublemma, and hence the lemma, follow. \square

Lemma 2.2. *Let $w = \sigma c = \sigma C_1^{\epsilon_1} \cdots C_d^{\epsilon_d} \in W(d)^o$ be a monomial. Then*

$$jtr_{C_i w C_i}(x_1, \dots, x_d) = \pm jtr_w(x_1, \dots, Jx_i J, \dots, x_d),$$

where the sign is $(-1)^{\sum \epsilon_j}$.

Proof. There are three slightly different cases depending on whether $\sigma(i)$ is less than, equal to, or greater than i . Trusting the reader to do the other two cases, we assume that $\sigma(i) < i$. In this case

$$C_i \sigma c C_i = \sigma C_{\sigma(i)} c C_i = g_1 \sigma c',$$

where

$$c' = C_1^{\epsilon_1} \cdots C_{\sigma(i)}^{\epsilon_1 + \sigma(i)} \cdots C_i^{1 + \epsilon_i} \cdots C_d^{\epsilon_d}$$

and where $g_1 = (-1)^{\gamma_1}$ and

$$\gamma_1 = \epsilon_1 + \cdots + \epsilon_{\sigma(i)-1} + \epsilon_{i+1} + \cdots + \epsilon_d.$$

To compute $jtr_{\sigma c'}$ we choose e_j with degrees ϵ_j , unless $j = i$ or $\sigma(i)$ in which case the degree is one greater. Now

$$e_d \cdots e_1 jtr_{\sigma c'}(x_1, \dots, x_d) = tr_{\sigma}(e_1 J^{\epsilon_1} x_1, \dots, e_d J^{\epsilon_d} x_d). \quad (9)$$

The i -th term is $C_i J^{1+\epsilon_i} x_i$ and the $\sigma(i)$ term is $e_{\sigma(i)} J^{1+\epsilon_{\sigma(i)}} x_{\sigma(i)}$. We can assume without loss of generality that $e_{\sigma(i)}$ factors as $f_1 f_2$, where f_1 is of degree 1 and f_2 is of degree $\epsilon_{\sigma(i)}$. An important but elementary fact about trace monomials is that if $y_{\sigma(i)} = ab$ then $\text{tr}_\sigma(y_1, \dots, y_d) = \text{tr}_\sigma(y_1, \dots, y_i a, \dots, b, \dots, y_d)$, where in the right hand side of the equation y_i is replaced by $y_i a$ and $y_{\sigma(i)}$ is replaced by b . In order to apply this to (9) we first note that $y_{\sigma(i)} = f_1 f_2 J^{1+\epsilon_{\sigma(i)}} x_{\sigma(i)}$ which we re-write as $(-1)^{\epsilon_{\sigma(i)}} f_1 J f_2 J^{\epsilon_{\sigma(i)}} x_{\sigma(i)}$. For the sake of bookkeeping, we let $g_2 = (-1)^{\gamma_2}$ where $\gamma_2 = \epsilon_{\sigma(i)}$. Then (9) equals

$$g_2 \cdot \text{tr}_\sigma(e_1 J^{\epsilon_1} x_1, \dots, f_2 J^{\epsilon_{\sigma(i)}} x_{\sigma(i)}, \dots, e_i J^{1+\epsilon_i} x_i f_1 J, \dots).$$

Since f_1 is degree one and supercentral, the i^{th} argument equals $f_1 e_i J^{\epsilon_i} J x_i J$, and since $f_1 e_i$ has degree ϵ_i this equals

$$g_2 \cdot e_d \cdots e_{i+1} f_1 e_i e_{i-1} \cdots e_{\sigma(i)+1} f_2 e_{\sigma(i)-1} \cdots e_1 j \text{tr}_w(x_1, \dots, J x_i J, \dots, x_n)$$

and to complete the proof we need only pull the f_1 back to be in front of the f_2 and compute the sign. First,

$$e_d \cdots f_1 \cdots f_2 e_i \cdots e_1 = g_3 \cdot e_d \cdots e_1$$

where $g_3 = (-1)^{\gamma_3}$ and $\gamma_3 = \epsilon_{\sigma(i)+1} \cdots + \epsilon_i$. Altogether the sign is $g_1 g_2 g_3 = (-1)^{\gamma_1 + \gamma_2 + \gamma_3}$ and $\gamma_1 + \gamma_2 + \gamma_3 = \epsilon_1 + \cdots + \epsilon_d$, which proves the lemma. \square

2.3 Multilinear Identities

Given a monomial $c = C_1^{\epsilon_1} \cdots C_d^{\epsilon_d}$ in $C(d)^o$ we define $|c|$ to be $\sum \epsilon_j$ in \mathbb{Z}_2 , and we sometimes refer to c as even or odd depending on whether $|c|$ is 0 or 1. Likewise, for $\sigma c \in W(d)^o$ we define $|\sigma c|$ to be $|c|$.

Lemma 2.3. *Let A be an algebra with a J -trace. Assume that*

$$u = \sum_{\sigma, \epsilon} \alpha(\sigma, \epsilon) \sigma c^\epsilon \in W(d)$$

is such that $j \text{tr}_u$ identity of A , and write u as $u = u_0 + u_1$, the even terms plus the odd terms. Then $j \text{tr}_{u_0}$ and $j \text{tr}_{u_1}$ are each identities for A .

Proof. Given any $a_1, \dots, a_d \in A_0$,

$$j \text{tr}_u(a_1, \dots, a_d) = j \text{tr}_{u_0}(a_1, \dots, a_d) + j \text{tr}_{u_1}(a_1, \dots, a_d) = 0.$$

But since the first term is in A_0 and the second in A_1 each must be zero. \square

Combining this lemma with lemmas 2.1 and 2.2 of the previous section we get this theorem.

Theorem 2.4. *Let A be a J -trace algebra and let jtr_f be an identity for A for some $f \in W(d)^o$. Let $w = \sigma c$ be a monomial in $W(d)^o$. Then $jtr_{wf w^{-1}}$ is also an identity for A .* \square

\square

The correspondence $w \leftrightarrow jtr_w$ is a vector space isomorphism between $W(d)^o$ and the degree d , multilinear, pure J -trace polynomials in x_1, \dots, x_d , which we denote JTR_d . Using this identification we may consider JTR_d as a bimodule for $W(d)^o$. Also note that the inclusions $W(d)^o \subseteq W(e)^o$ for $d \leq e$ correspond to the inclusions $JTR_d \subseteq JTR_{d+1}$ gotten by corresponding $f(x_1, \dots, x_d)$ to $f(x_1, \dots, x_d)str(x_{d+1}) \cdots str(x_e)$. For a fixed J -trace-algebra A we let $I_d \subseteq JTR_d$ be the set of identities for A in JTR_d .

Convention. Theorem 2.4 shows that I_d is invariant under conjugation by monomials in $W(d)^o$ but in general it need not be invariant under right or left multiplication. For the remainder of this section we will consider the special case in which I_d is both a left and right $W(d)^o$ module and we will prove that the elements of $W(e)I_dW(e)$ are all consequences of I_d and hence are in I_e , for all $d \leq e$.

We remark that the corresponding statement is a key lemma in the derivations of the trace identities for matrices and for $M_{k,\ell}$ by Razmyslov, Procesi and Berele mentioned previously.

First of all, by induction, it suffices to prove the statement for $e = d + 1$. Next, it suffices to prove that if $f \in I_d$ then fw , $wf \in I_{d+1}$ for each monomial $w \in W(d + 1)^o$, and by Theorem 2.4 we need only consider the case of wf .

Lemma 2.5. *Let $\sigma \in S_{d+1}$ fix $d + 1$. Then $tr_{\sigma(i,d+1)}(x_1, \dots, x_{d+1}) = tr_{\sigma}(x_1, \dots, x_{d+1}x_i, \dots, x_d)$.*

Proof. The proof follows from the computation

$$(i, a, \dots, b)(i, d + 1) = (i, d + 1, a, \dots, b)$$

together with equation (6). \square

Lemma 2.6. *Let $f = jtr_w$ for some $w \in I_d$. Then for each $u \in W(d + 1)^o$, $uf \equiv jtr_{uw}$ is a consequence of I_n .*

Proof. By the previous lemma we may assume without loss of generality that either all terms of w are even or all terms are odd.

Let $u = \sigma c$, $\sigma \in S_{d+1}$, $c \in C(d+1)$. We first consider the case of $\sigma \in S_d$ so that $\sigma(d+1) = d+1$. In this case u can be re-written as $C_{d+1}^{\epsilon_{d+1}} u'$ where $u' \in C(d)$ and so $u' f \in I_d$. Hence, in this case, we may replace f by $u' f$ and so assume without loss that $u = C_{d+1}^{\epsilon_{d+1}}$. Also, by Lemma 2.3 we may assume that every term in w is of degree $g \bmod 2$. In this case

$$\begin{aligned} jtr_{C_{d+1}^{\epsilon_{d+1}} w}(x_1, \dots, x_{d+1}) &= (-1)^{g_{\epsilon_{d+1}}} jtr_{w C_{d+1}^{\epsilon_{d+1}}}(x_1, \dots, x_{d+1}) \\ &= (-1)^{g_{\epsilon_{d+1}}} jtr_w(x_1, \dots, x_d) str(J^{\epsilon_{d+1}} x_{d+1}), \end{aligned}$$

which is certainly a consequence of f .

If $w = \sigma c$ with $\sigma \notin S_d$ we may write σ as $(i, d+1)\sigma'$, where $\sigma'(d+1) = d+1$; and we write c as $c = C_{d+1}^{\epsilon_{d+1}} c'$ where $c' \in C(d)$. Then C_{d+1} commutes with σ' and so $w = (i, d+1)C_{d+1}^{\epsilon_{d+1}} w'$ where $w' \in W(d)^o$, and since $W(d)^o I_n \subseteq I_n$ it suffices to consider the case in which $w = (i, d+1)C_{d+1}^{\epsilon_{d+1}}$. We now compute the J -trace polynomial corresponding to wu where $w = (i, d+1)C_{d+1}^{\epsilon_{d+1}}$ and u is a monomial $u = \sigma c = \sigma C_1^{\epsilon_1} \dots C_d^{\epsilon_d}$. First note that $wu = (-1)^\gamma (i, d+1) \cdot u C_{d+1}^{\epsilon_{d+1}}$, where γ equals $\epsilon_{d+1}(\epsilon_1 + \dots + \epsilon_d)$; and since the multiplication of the permutations in $W(d+1)^o$ is in the opposite their product in S_{d+1} , wu equals $(-1)^\gamma \sigma(i, d+1) C_{d+1}^{\epsilon_{d+1}}$. Let e_i have degree ϵ_i . Then by (8)

$$e_{d+1} \dots e_1 jtr_{wu} = (-1)^\gamma \cdot tr_{\sigma(i, d+1)}(e_1 J^{\epsilon_1} x_1, \dots, e_{d+1} J^{\epsilon_{d+1}} x_{d+1}). \quad (10)$$

Applying Lemma 2.5, (10) equals

$$(-1)^\gamma \cdot tr_\sigma(e_1 J^{\epsilon_1} x_1, \dots, e_i J^{\epsilon_i} x_i e_{d+1} J^{\epsilon_{d+1}} x_{d+1}, \dots, e_d J^{\epsilon_d} x_d).$$

Focusing on the i^{th} term:

$$e_i J^{\epsilon_i} x_i e_{d+1} J^{\epsilon_{d+1}} x_{d+1} = e_{d+1} e_i J^{\epsilon_i + \epsilon_{d+1}} J^{\epsilon_{d+1}} x_{d+1} J^{\epsilon_{d+1}} x_{d+1}$$

and so (10) equals

$$(-1)^\gamma \cdot e_d \dots e_{i+1} e_{d+1} e_i \dots e_1 jtr_{\sigma c'}(x_1, \dots, J^{\epsilon_{d+1}} x_i J^{\epsilon_{d+1}} x_{d+1}, \dots, x_d),$$

where $c' = C_1^{\epsilon_1} \dots C_i^{\epsilon_i + \epsilon_{d+1}} \dots C_d^{\epsilon_d}$. The above equals

$$(-1)^\gamma \cdot e_{d+1} \dots e_1 jtr_{u C_i^{\epsilon_i + \epsilon_{d+1}}}(x_1, \dots, J^{\epsilon_{d+1}} x_i J^{\epsilon_{d+1}} x_{d+1}, \dots, x_d).$$

and so

$$jtr_{wu} = (-1)^\gamma \cdot jtr_{uC_i^{\epsilon_{d+1}}}(x_1, \dots, J^{\epsilon_{d+1}}x_i J^{\epsilon_{d+1}}x_{d+1}, \dots, x_d).$$

Since $uC_i^{\epsilon_{d+1}} \in W(d)^o$ and the above polynomial is gotten from $jtr_{uC_i^{\epsilon_{d+1}}}$ by the homogeneous substitution $x_i \mapsto J^{\epsilon_{d+1}}x_i J^{\epsilon_{d+1}}x_{d+1}$ it is a consequence of it. \square

Theorem 2.7. *Assume that $W(d)^o I_d W(d)^o = I_d$. Then for all $e \geq d$ the elements of $W(e)^o I_d W(e)^o$ are all consequences of I_d . In particular, they are all identities of A .*

Proof. Follows from the previous lemma by induction on e . \square

3 J -trace-Identities of $M(n, n)$

3.1 Theorems of Sergeev and Berele

For the remainder of the paper we take F to be characteristic zero. Let U be a \mathbb{Z}_2 -graded vector space with graded dimension (n, n) and assume that J is a degree 1 idempotent isomorphism of U , i.e., a linear transformation with $J^2 = id$, $J(U_0) = U_1$ and $J(U_1) = U_0$. In [9] Sergeev defined an action of $W(d)^o$ on $U^{\otimes d}$ such that if $u_1, \dots, u_d \in U$ are homogeneous, then

$$\sigma(u_1 \otimes \dots \otimes u_d) = \pm u_{\sigma(1)} \otimes \dots \otimes u_{\sigma(d)}$$

and

$$C_i(u_1 \otimes \dots \otimes u_d) = \pm u_1 \otimes \dots \otimes Ju_i \otimes \dots \otimes u_d,$$

where the sign depends on how many degree one elements are switched. The exact details are not necessary for our purposes. Sergeev proved two basic properties of this action, one about the image of $W(d)^o$ and one about the kernel of the action. The former result is that the image of $W(d)^o$ in $End(U^{\otimes d})$ equals the centralizer of the action of $q(n)$, the queer superalgebra. For the kernel, it is important to know that $W(d)^o$ is a direct sum of two sided ideals $\sum \oplus I_\lambda$, each I_λ is a simple graded algebra, and the subscripts are naturally indexed by partitions of d into distinct parts. If we write $\phi : W(d)^o \rightarrow End(U^{\otimes d})$ then Sergeev proved that the kernel is the sum $\sum \oplus I_\lambda$ summed over all partitions of height greater than n . Note that the smallest such partition would be $\delta = (n+1, n, \dots, 1)$ and in general if λ is a partition with distinct parts, λ will have height greater than n if and only if $\delta \subseteq \lambda$. For convenience, we record these results of Sergeev.

Theorem 3.1. *The algebra $W(d)^o$ is semisimple as an algebra and as a graded algebra. As a graded algebra it decomposes as a direct sum of graded simple ideals $\sum \oplus I_\lambda$ indexed by partitions of d with all distinct parts. There is a natural map $\phi : W(d)^o \rightarrow \text{End}(V^{\otimes d})$ whose kernel is the sum of the I_λ where λ has height greater than d .*

Definition 3.2. We let $DP(d)$ be the set of partitions of d into distinct parts.

In [2] we let V be a $2n$ dimensional space over the Grassmann algebra E . The algebra $\text{End}_E(V)$ is isomorphic to $M(n, n)$ as a superalgebra with supertrace. Using the notion of extending an F -linear map to an E linear one, we embed $\text{End}(U^{\otimes d})$ into $\text{End}(V^{\otimes d})$. Composing this with ϕ creates a map

$$\Phi : W(d)^o \rightarrow \text{End}(V^{\otimes d})$$

which has the same kernel as ϕ . Next, adapting some standard techniques from invariant theory we constructed an isomorphism

$$\text{End}(V^{\otimes d}) \cong (\text{End}(V)^{\otimes d})^*.$$

We denote by T_w the functional on $\text{End}(V)^{\otimes d}$ corresponding to $w \in W(d)^o$.

In section 5 of [2] we computed $T_w(A_1 \otimes \cdots \otimes A_d)$ for A_1, \dots, A_d degree zero elements of $\text{End}(V)$. Let $w = \sigma C_1^{\epsilon_1} \cdots C_d^{\epsilon_d} \in W(d)^o$. In lemma 5.8 we showed that

$$T_w(A_1 \otimes \cdots \otimes A_d) = T_\sigma(\tilde{P}^{\epsilon_1} A_1 \otimes \cdots \otimes \tilde{P}^{\epsilon_d} A_d),$$

where \tilde{P} is a degree one element of $\text{End}(V)$ corresponding to J . Letting e_i be Grassmann elements of degree ϵ_i we showed in the proof of lemma 5.7 that

$$e_d \cdots e_1 T_\sigma(\tilde{P}^{\epsilon_1} A_1 \otimes \cdots \otimes \tilde{P}^{\epsilon_d} A_d) = T_\sigma(e_1 \tilde{P}^{\epsilon_1} A_1 \otimes \cdots \otimes e_d \tilde{P}^{\epsilon_d} A_d).$$

Continuing to work backwards through the lemmas of [2], lemma 5.6 can be used to evaluate the right hand side of this equation, since the factors in the tensor product are now all degree zero. It implies that

$$T_\sigma(e_1 \tilde{P}^{\epsilon_1} A_1 \otimes \cdots \otimes e_d \tilde{P}^{\epsilon_d} A_d) = \text{tr}_\sigma(e_1 \tilde{P}^{\epsilon_1} A_1, \dots, e_d \tilde{P}^{\epsilon_d} A_d).$$

Combining all this we get

$$e_d \cdots e_1 T_w(A_1 \otimes \cdots \otimes A_d) = \text{tr}_\sigma(e_1 \tilde{P}^{\epsilon_1} A_1, \dots, e_d \tilde{P}^{\epsilon_d} A_d),$$

and comparing with (8) gives this lemma:

Lemma 3.3. *If $A_1, \dots, A_d \in \text{End}(V)_0$ and $w \in W(d)^o$ then $T_w(A_1 \otimes \dots \otimes A_d) = jtr_w(A_1, \dots, A_d)$. \square*

It follows that $w \in W(d)^o$ corresponds to a J -trace-identity for $\text{End}(V)$ if and only if T_w is zero. But $T_w = 0$ precisely when w is in the kernel of Φ , which is the same as the kernel of ϕ and which was identified by Sergeev. Finally, $\text{End}(V)_0$ is isomorphic to $M(n, n)$ as an algebra with J -trace. We now have the main theorem of this section:

Theorem 3.4. *Let $w \in W(d)^o$. Then jtr_w is a J -trace-identity for $M(n, n)$ if and only if w lies in the two sided ideal $\sum \oplus I_\lambda$, summed over all λ of height greater than n . \square*

3.2 A Combinatorial Lemma

Theorem 3.4 tells us that the multilinear, degree d identities of $M(n, n)$, $I_d = I_d(M(n, n))$, form a two sided ideal in each $W(d)^o$; and Theorem 2.7 says that the two sided ideal induced from I_d , considered as a subspace of $W(d)^o$, to each $W(e)^o$, $e > d$ consists of algebraic consequences of I_d . Fortunately, the question of how ideals in $W(d)^o$ induce up is easily handled by a theorem of Schur and Jozefiak, and a theorem of Stembridge, as we now relate. We first focus on the case of $W(d+1)^o I_\lambda W(d+1)^o$, where λ is a partition of d in $DP(d)$ so I_λ is an ideal of $W(d)^o$.

Lemma 3.5. *If $I \triangleleft W(d)^o$ is a two sided ideal, and if $W(d+1)^o I$ decomposes as a direct sum of left ideals as $\sum m_\mu M_\mu$, where $M_\mu \subseteq I_\mu$ is a minimal left ideal and $m_\mu \neq 0$, then $W(d+1)^o I W(d+1)^o = \sum I_\mu$ summed over all μ with $m_\mu \neq 0$.*

Proof. Since M_μ is a right ideal contained in the minimal two-sided ideal I_μ ,

$$M_\mu I_\nu = \begin{cases} 0, & \mu \neq \nu \\ I_\mu, & \mu = \nu \end{cases}$$

\square

The computation of $W(d+1)^o I_\lambda W(d+1)^o$ is accomplished in the next lemma.

Lemma 3.6. *Let $\lambda \in DP(d)$. Then $W(d+1)^o I_\lambda W(d+1)^o = \sum \oplus I_\mu$ where μ runs over all partitions in $DP(d+1)$ containing λ .*

Proof. As a right $W(d)^o$ module I_λ decomposes as a direct sum of isomorphic simple modules M_λ . Now

$$W(d+1)^o M_\lambda \cong W(d+1) \otimes_{W(d)^o} M_\lambda \cong W(d+1)^o / W(d)^o \otimes_F M_\lambda.$$

We compare this to

$$M_{[1]} \hat{\otimes} M_\lambda = W(d+1)^o \otimes_{W(1)^o \times W(d)^o} (M_{[1]} \otimes_F M_\lambda).$$

However, $W(1)^o$ is graded simple of dimension $(1, 1)$ and $M_{[1]}$ is isomorphic to $W(1)^o$. It follows that the right hand side is isomorphic to

$$W(d+1)^o \otimes_{W(d)^o} M_\lambda$$

and so $W(d+1)^o M_\lambda$ is isomorphic to $M_{[1]} \hat{\otimes} M_\lambda$. The outer tensor product now follows from the theorems we alluded to earlier. First, just like the outer tensor product of modules for the symmetric group is reflected in the product of Schur functions, Schur and Jozefiak proved that the outer product of Sergeev modules is reflected in the product of Q-Schur functions. See [5] for an account of these functions.

Theorem (Schur, Jozefiak). *The Grothendick ring of $W(d)^o$ modules is isomorphic to the ring of Q-Schur functions with the isomorphism given by $M_\lambda \mapsto Q_\lambda(x)$.*

This means that $M_\mu \hat{\otimes} M_\nu = \sum m(\lambda; \mu, \nu) M_\lambda$ if and only if $Q_\mu(x)Q_\nu(x) = \sum m(\lambda; \mu, \nu) Q_\lambda(x)$. The latter products were computed by Stembridge in [10], see also [5] (8.18). Stembridge proved an analogue of the Littlewood-Richardson rule for the functions $P_\lambda(x)$ which have the property that each $Q_\lambda(x)$ is a (non-zero) constant multiple of $P_\lambda(x)$ and so we can use Stembridge's rule to determine when the coefficients we are calling $m(\lambda; \mu, \nu)$ are non-zero. We will need only the following special case:

Theorem (Stembridge). *Let $\lambda \in DP(n)$ and $\mu \in DP(n+1)$. Then $m(\mu; \lambda, [1]) \neq 0$ if and only if $\lambda \subseteq \mu$.*

Our lemma now follows. □

The following corollary follows by induction.

Corollary 3.7. *Let δ be the partition $(n+1, n, \dots, 1) \in DP(\binom{n+2}{2})$ and let $e > \binom{n+2}{2}$. Then $W(e)^o I_\delta W(e)^o$ equals the sum $\sum \oplus I_\lambda$, where λ runs over partitions of e with distinct parts, of height greater than n .*

The Main Theorem

Theorem 2.7, Theorem 3.4 and Corollary 3.7 are the ingredients we need to prove our main theorem.

Theorem 3.8. *All pure J -trace identities of $M(n, n)$ are consequences of the identities of degree $\binom{n+2}{2}$.*

Proof. By theorem 3.4 there are no J -trace identities of degree less than $d = \binom{n+2}{2}$, and in degree $\binom{n+2}{2}$ the multilinear identities correspond to the two sided ideal of $W(d)^o$, I_δ . By theorem 2.7 for each $e > d$ all elements of $W(e)^o I_\delta W(e)^o$ are consequences of I_δ . By corollary 3.7 the product $W(e)^o I_\delta W(e)^o$ equals the sum of the two sided ideals I_λ where λ has height greater than n . Finally, by theorem 3.4 again, this sum is precisely the space of multilinear identities of degree e , and since we are in characteristic zero all identities are consequences of multilinear ones and the theorem follows. \square

Using the following lemma we can also describe the mixed J -trace-identities of $M(n, n)$.

Lemma 3.9. *Let $f(x_1, \dots, x_{n+1}) = \sum_\alpha \text{str}(x_{n+1} u_\alpha) g_\alpha$ be a multilinear pure J -trace-identity for $M(n, n)$, where each u_α is a polynomial (without trace) in $\{J, x_1, \dots, x_n\}$ and each g_α is a pure J -trace-polynomial. Then $F(x_1, \dots, x_n) = \sum_\alpha u_\alpha g_\alpha$ is a multilinear mixed J -trace-identity for $M(n, n)$ of degree one smaller and f is a consequence of F .*

Proof. Since the supertrace is E -linear we have $f = \text{str}(x_{n+1} \sum u_\alpha g_\alpha)$, and the lemma follows from the non-degeneracy of the supertrace. \square

The lemma and theorem combine to give this description of the J -trace-identities of $M(n, n)$.

Theorem 3.10. *All J -trace identities of $M(n, n)$ are consequences of the identities of degree $\binom{n+2}{2} - 1$.*

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